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Applications of Volterra's theory of composition to hypergeometric functions

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APPLICATIONS OF VOLTERRA'S THEORY OF COMPOSITION TO HYPERGEOMETRIC FUNCTIONS

by

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I. INTRODUCTION

The problem of this dissertation is to derive, if possible, new properties for some of the important functions which occur in the mathematical literature. The method used for the derivation of most of these properties consists of an application of Volterra's theory of composition. Some of the functions considered are the Tchebycheff polynomials, the Bessel functions, and the Laguerre polynomials, all of which are particular cases of the generalized hypergeometric function given by equation (3.1).

Among the properties of these functions which will be considered are integral addition theorems. Volterra[1] (18) has shown in the theory of composition that integral addition theorems can be obtained from algebraic addition formulas. He accomplished this by means of the isomorphism which exists between algebraic formulas which involve only addition and multiplication and those obtained from the algebraic formulas by replacing powers of the variable by powers by composition[2] of a function $f(x,y)$. Volterra used the algebraic addition formula of the function $v(a,z) = e^{az}-1$, namely,

[1]References to the bibliography are made by numbers in parentheses; the specific page number is given when necessary.

[2]For the definition of composition and the subsequent notation used see Definition 2.1 and Definition 2.2.

$$v(a,z)\ v(b,z) = v(a+b,z) - v(a,z) - v(b,z),$$

in order to derive the integral addition theorem

$$V(a,f^*)^*\ V(b,f^*) = V(a+b,f^*) - V(a,f^*) - V(b,f^*).$$

The function $V(a,f^*)$ is called the Volterra transcendental. If it is possible to express a function as a Volterra transcendental, then the above integral addition theorem follows for the function immediately.

New integral addition theorems can then be obtained either by giving new expressions for certain functions in terms of the Volterra transcendental or by finding algebraic addition formulas from which new integral addition theorems can be derived. The method used here to solve the latter problem is to define a function analogous to the Volterra transcendental, namely, the "Tchebycheff transcendental," and to show that it satisfies a certain integral addition theorem.

As a further application of the theory of composition certain integrals involving products of hypergeometric functions are evaluated by taking the Volterra transform (see Definition 3.1) of both members of equations involving products of hypergeometric functions. Also, expansions of certain functions are derived by taking the Volterra transform of known series expansions of related functions.

Throughout the thesis some minor problems, which arose in connection with the more general investigation, are considered.

Some of these are connected with the theory of composition and some deal with special properties of some of the hypergeometric functions. In particular, section C of Chapter IV was prompted by a recent note on a series of products of Legendre polynomials (16).

Most of the literature concerning the theory of composition can be found in the various works of Volterra and Peres, namely, (9), (17), (18), (19), with (18) being the most extensive treatise on the subject.

II. VOLTERRA'S THEORY OF COMPOSITION

A. Fundamental Definitions and Theorems

Let $f(x,y)$ and $g(x,y)$ be two functions such that the integral

$$\int_x^y f(x,t)\, g(t,y)dt \tag{2.1}$$

exists for $a \leq x \leq y \leq b$.

Definition 2.1. The integral (2.1) is called the product by composition of $f(x,y)$ and $g(x,y)$ and is denoted by f^*g.

This type of composition is generally known as that of the first kind. Volterra (19, p. 99) has defined a second kind of composition in which the limits of integration are constants.

The following well known theorem can easily be proved with the use of Dirichlet's formula for interchanging the order of integration (18, p. 6).

Theorem 2.1. The operation of composition is associative, that is,

$$f^*(g^*h) = (f^*g)^*h,$$

where $h = h(x,y)$.

It follows from the properties of integration that composition is also distributive, that is,

$$f^*(g+h) = f^*g+f^*h$$

$$(g+h)^*f = g^*f+h^*f.$$

Definition 2.2. If $f^*g = g^*f$ then the two functions f and g are said to be permutable.

Definition 2.3. If f is composed with itself, the resulting function $f^*f = f^{*2}$ is called the square by composition of f; and in general the function $f^{*n} = f^{*(n-1)*}f$, n an integer ≥ 2, is called the nth power by composition of f. The symbol f^{*1} is defined as equal to f.

For the powers by composition of f, the following theorem holds (19, p. 101). It is a direct consequence of the associative law.

Theorem 2.2. *The powers by composition of a function are permutable with one another, that is,*

$$f^{*n*}\, f^{*m} = f^{*m*}\, f^{*n} = f^{*m+n},$$

where m and n are positive integers.

Definition 2.4. If one considers an aggregate of permutable functions $f_1, f_2, \ldots, f_i$, and forms the sum of a finite number of terms of the type

$$b\, f_1^{*n_1}\, f_2^{*n_2} \ldots f_i^{*n_i},$$

where $n_1, n_2, \ldots, n_i$ are positive integers and b is a real constant, then the expression so obtained will be called a polynomial by composition.

For these polynomials by composition the following theorem holds (18, p. 7).

Theorem 2.3. A polynomial by composition constructed with permutable functions $f_1, f_2, \ldots, f_i$ is a new function which is itself permutable with these functions; the composition of the polynomials will be carried out by the same rules as those that hold for the product of ordinary polynomials of the variables $f_1, f_2, \ldots, f_i$.

B. The Algebra of Permutable Functions

Evans (5) has set up a set of postulates which defines the algebra of permutable functions. The postulates of addition are the following.

(1) $f+g$ exists in the system.

(2) $(f+g)+h = f+(g+h)$.

(3) If $f+g = f+h$, then $g = h$.
 If $g+f = h+f$, then $g = h$.

(4) If $bf = bg$, $b \neq 0$, then $f = g$.

The postulates of multiplication (composition) are the following.

(1) $f^* g$ exists in the system.

(2) $(f^* g)^* h = f^*(g^* h)$.

(3) If $f^* g = f^* h$ and $f \neq 0$, then $g = h$.

If $g^* f = h^* f$ and $f \neq 0$, then $g = h$.

(4) $f^*(g+h) = f^* g + f^* h$.

$(g+h)^* f = g^* f + h^* f$.

(5) $g^* f = f^* g$.

Since these postulates correspond to the postulates of ordinary algebra, it follows that the identities of the latter, in so far as they depend upon these postulates, have their analogues in the algebra of permutable functions.

It is now possible to pass from polynomials by composition to the consideration of more general functional operations, obtained from ordinary power series by substituting powers by composition of a function f for ordinary powers. For example, from the power series $\sum_1^\infty z^i$ convergent within the circle $|z| < 1$, a corresponding series of composition $\sum_1^\infty f^{*i}$ can be obtained, which for any bounded function f will be convergent everywhere. For, if $|f| < L$,

$$|f^{*n}| < \frac{L^n (y-x)^{n-1}}{(n-1)!},$$

that is, the terms of the series of composition are in absolute value less than the terms of an exponential series. The following theorem can be proved (19, p. 103).

<u>Theorem</u> 2.4. <u>If</u>

$$F(z_1,z_2,\ldots,z_n) = \sum_1^{\infty} i_1 i_2 \ldots i_n a_{i_1 i_2 \ldots i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}$$

<u>is a power series in the variables</u> z_r <u>which is convergent for sufficiently small</u> $|z_r|$'s $(r = 1,2,\ldots,n)$, <u>then the series</u>

$$F(f_1^*,f_2^*,\ldots,f_n^*) = \sum_1^{\infty} i_1 i_2 \ldots i_n\, a_{i_1 i_2 \ldots i_n} f_1^{*i_1} f_2^{*i_2} \ldots f_n^{*i_n}$$

<u>is convergent provided the</u> f_r's <u>are bounded, and this series represents a function of</u> x <u>and</u> y <u>which is permutable with all the</u> f_r's <u>if these are permutable with each other</u>.

Thus there is an isomorphism between algebraic formulas which involve only addition and multiplication and those obtained from the algebraic formulas by replacing the powers of the variables with powers by composition of the corresponding functions f_i. It follows that if $F(z_1,z_2,\ldots,z_n)$ and $G(z_1,z_2,\ldots,z_n)$ are two analytic functions expressible as power series involving only positive integral powers and if

$$F(z_1,z_2,\ldots,z_n)\, G(z_1,z_2,\ldots,z_n) = K(z_1,z_2,\ldots,z_n),$$

then

$$F(f_1^*,f_2^*,\ldots,f_n^*)^* G(f_1^*,f_2^*,\ldots,f_n^*) = K(f_1^*,f_2^*,\ldots,f_n^*).$$

In particular by this isomorphism addition formulas in ordinary algebra give rise to integral addition formulas. For example, since the function $v(a,z) = e^{az}-1$ satisfies the algebraic addition formula

$$v(a,z)\ v(b,z) = v(a+b,z)-v(a,z)-v(b,z), \qquad (2.2)$$

as can be easily verified, it follows that the replacement of powers of z by powers by composition of f leads to the integral addition theorem

$$V(a,f^*)^*\ V(b,f^*) = V(a+b,f^*)-V(a,f^*)-V(b,f^*). \qquad (2.3)$$

The function $V(a,f^*) = \sum_{n=1}^{\infty} \frac{a^n}{n!} f^{*n}(x,y)$ is called the Volterra transcendental.

In general it may be quite difficult to establish integral addition theorems for a given function. If, however, it can be expressed as a Volterra transcendental then the convergence of its expansion and the validity of the integral addition theorem given by equation (2.3) follow at once.

C. Integral Addition Theorems Obtained by the Use of Volterra's Transcendental

In order to apply the properties of the Volterra transcendental to important classes of functions occurring in the mathematical literature, it is necessary to find the nth power by composition of certain functions. In particular, if $f(x,y)=1$, then

$$1^{*n} = \frac{(y-x)^{n-1}}{(n-1)!}, \qquad (2.4)$$

a result which can be easily verified by mathematical induction. In analogy with equation (2.4), 1^{*a}, $a>0$, is defined as

$$\frac{(y-x)^{a-1}}{\Gamma(a)}.$$

The composition of 1^{*a} with 1^{*b}, $b>0$, is

$$1^{*a} {}^{*}1^{*b} = \int_x^y \frac{(t-x)^{a-1}}{\Gamma(a)} \frac{(y-t)^{b-1}}{\Gamma(b)}\,dt$$

which is equal to

$$\frac{(y-x)^{a+b-1}}{\Gamma(a+b)}.$$

Thus the following law is established for all $a>0$, $b>0$:

$$1^{*a} {}^{*}1^{*b} = 1^{*a+b}. \qquad (2.5)$$

If in the integral addition formula (2.3), $f(x,y) = 1$, then one obtains an integral addition theorem involving Bessel functions of the first order, a result which has been given previously in the literature (12). More generally, however, if $f(x,y) = \frac{(y-x)^{p-1}}{\Gamma(p)}$ for p an integer >0, one obtains the following integral addition theorem for the hypergeometric series ${}_0F_p$.

Theorem 2.5.

$$\int_x^y \frac{[(t-x)(y-t)]^{p-1}}{\Gamma(p)\ \Gamma(p)}\ {}_0F_p\left[2,\frac{p+1}{p},\ldots,\frac{2p-1}{p};a\frac{(t-x)^p}{p^p}\right]$$

$$\cdot\ {}_0F_p\left[2,\frac{p+1}{p},\ldots,\frac{2p-1}{p};b\frac{(y-t)^p}{p^p}\right]dt$$

$$=\frac{(a+b)}{ab}\ \frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_0F_p\left[2,\frac{p+1}{p},\frac{p+2}{p},\ldots,\frac{2p-1}{p};(a+b)\ \frac{(y-x)^p}{p^p}\right]$$

$$-\frac{(y-x)^{p-1}}{b\ \Gamma(p)}\ {}_0F_p\left[2,\frac{p+1}{p},\frac{p+2}{p},\ldots,\frac{2p-1}{p};a\frac{(y-x)^p}{p^p}\right]$$

$$-\frac{(y-x)^{p-1}}{a\ \Gamma(p)}\ {}_0F_p\left[2,\frac{p+1}{p},\frac{p+2}{p},\ldots,\frac{2p-1}{p};b\frac{(y-x)^p}{p^p}\right].$$

Proof: If $f(x,y)=\dfrac{(y-x)^{p-1}}{\Gamma(p)}=1^{*p}$, the Volterra transcendental $V(a,f^*)$ becomes

$$\sum_{n=1}^{\infty}\frac{a^n}{n!}\ \frac{(y-x)^{pn-1}}{\Gamma(pn)},$$

which in turn is equal to

$$a\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_0F_p\left[2,\frac{p+1}{p},\frac{p+2}{p},\ldots,\frac{2p-1}{p};a\frac{(y-x)^p}{p^p}\right].$$

The desired result is then obtained if one sets $f(x,y)=1^{*p}$ in the integral addition formula (2.3).

In particular, if one sets $p=2$, the above theorem yields an integral addition formula for the particular Humbert function (24)

$$J_{1,\frac{1}{2}}\left[-\frac{3a^{1/3}(y-x)^{2/3}}{\sqrt[3]{4}}\right],$$

since

$$J_{m,n}(x) = \left(\frac{x}{3}\right)^{m+n} \frac{1}{\Gamma(m+1)\Gamma(n+1)} \, {}_0F_2\left[m+1, n+1; -\frac{x^3}{27}\right].$$

If in the integral addition formula (2.3), $f(x,y)$ is set equal to $\frac{(y-x)^{p-1}}{\Gamma(p)}$ for p greater than zero but not an integer, no such general integral addition theorem as Theorem 2.5 will result. All of the integral addition theorems obtained in this manner will be dependent upon the specific value assigned to p.

It might be well to point out here that it is quite difficult to obtain the nth power by composition of some rather simple functions. For example, if $f(x,y) = x+y$, it can be shown that

$$f^{*2} = \frac{(y-x)}{3!}(5x^2+14xy+5y^2)$$

$$f^{*3} = \frac{(y-x)^2}{5!}(5.9x^3+5.13.3x^2y+3.5.13xy^2+5.3^2y^3)$$

$$f^{*4} = \frac{(y-x)^3}{7!}(5.9.13x^4+2^2.3.281x^3y+2.3^2.307x^2y^2 +2^2.3.281xy^3+3^2.5.13y^4)$$

$$f^{*5} = \frac{(y-x)^4}{9!}(5.9.13.17x^5+3^2.5.19.83x^4y+2.3^2.5\,.\,1789x^3y^2 +2.3^2.5.1789x^2y^3+3^2.5.19.83xy^4+3^2.5.13.17y^5)$$

$$f^{*6} = \frac{(y-x)^5}{11!}(5.9.13.17.21x^6+2.3^2.5.7^2.401yx^5+5^2.3^2.31.733y^2x^4 +3^3.2^2.5^2.2633y^3x^3+5^2.3^2.31.733y^4x^2 +3^2.2.5.7^2.401y^5x+5.9.13.17.21y^6)\ .$$

By the investigation of these powers by composition, it is not apparent how f^{*n} can be expressed in terms of powers of x and y. Thus for the purposes at hand, the establishment of new integral addition theorems, it would be quite futile to set $f(x,y) = x+y$ in the equation (2.3).

The algebraic addition formula (2.3) can be generalized to the case of n parameters: $a_1, a_2, \ldots a_n$. This is done in the following manner. Rewriting equation (2.2) as

$$v(a_1,z)\, v(a_2,z) = v(a_1+a_2,z) - v(a_1,z) - v(a_2,z)$$

and multiplying both sides of this equation by $v(a_3,z)$, one obtains the following equation.

$$v(a_1,z)\, v(a_2,z) v(a_3,z) = v(a_1+a_2,z)\, v(a_3,z) - v(a_1,z)\, v(a_3,z) - v(a_2,z)\, v(a_3,z).$$

With the use of equation (2.2), the right-hand side of this equation can be rewritten as

$$v(a_1+a_2+a_3,z) - v(a_1+a_2,z) - v(a_1+a_3,z) - v(a_2+a_3,z) + v(a_1,z) + v(a_2,z) + v(a_3,z),$$

and thus

$$v(a_1,z)\, v(a_2,z)\, v(a_3,z) = v(a_1+a_2+a_3,z) - v(a_1+a_2,z) - v(a_1+a_3,z) - v(a_2+a_3,z) + v(a_1,z) + v(a_2,z) + v(a_3,z).$$

If one proceeds in a similar fashion and uses mathematical induction, the following general algebraic addition formula can be established

$$v(a_1,z)\ v(a_2,z)\ldots v(a_n,z) = v(a_1+\ldots+a_n,z)$$

$$-\sum_{i_1,\ldots,i_{n-1}} v(a_{i_1}+\ldots+a_{i_{n-1}},z)$$

$$+\ldots+(-1)^n \sum_{i_1,i_2,i_3} v(a_{i_1}+a_{i_2}+a_{i_3},z)$$

$$-(-1)^n \sum_{i_1,i_2} v(a_{i_1}+a_{i_2},z) + (-1)^n \sum_{i_1} v(a_{i_1},z),$$

where the summations refer to all combinations of the subscripts taken from the numbers 1, 2,..., n-1, n; one, two, three,..., and n-1 at a time. By replacing powers of z by powers by composition of f, one obtains the following generalization of the integral addition formula (2.3).

$$V(a_1,f^*)^*\ V(a_2,f^*)^*\ldots {}^*V(a_n,f^*) = V(a_1+\ldots+a_n,f^*)$$

$$-\sum_{i_1,\ldots i_{n-1}} V(a_{i_1}+\ldots+a_{i_{n-1}},f^*)+\ldots+$$

$$(-1)^n \sum_{i_1,i_2,i_3} V(a_{i_1}+a_{i_2}+a_{i_3},f^*) - (-1)^n \sum_{i_1,i_2} V(a_{i_1}+a_{i_2};f^*)$$

$$+ (-1)^n \sum_{i_1} V(a_{i_1},f^*).$$

If one sets $f(x,y)=1$ in this formula, a more general result is obtained than the integral addition theorem for Bessel functions given by Thielman (12). More generally, however, if $f(x,y)=1^{*p}$, p an integer >0, then the above integral addition formula will be a generalization of Theorem 2.5.

D. Functions Permutable with a Given Function

In all the considerations put forward up to now, it has been supposed that all the functions are permutable with each other. Thus a problem of special importance is that of determining the various sets of permutable functions for which there can exist a theory of composition. More precisely the fundamental problem is that of determining all the functions permutable with a given function. In this connection the following well known theorems are of importance. The proof of Theorem 2.6 is given by Volterra (18, p. 9) and the proof of Theorem 2.7 is due to Vessiot (15).

Theorem 2.6. *All analytic functions* $g(x,y)$ *permutable with a non-zero constant (which may be assumed to be unity), that is, all functions which satisfy the equation*

$$\int_x^y g(x,t)dt = \int_x^y g(t,y)dt$$

are functions of the difference $(y-x)$, *that is,* $g(x,y) = G(y-x)$.

Theorem 2.7. *All analytic functions permutable with a given analytic function are permutable with each other.*

Hence it follows that all functions of the difference (y-x) are permutable with one another and with unity. These functions form a particular set of permutable functions which Volterra has called the "group of the closed cycle." In much of the work on permutable functions, the group of the closed cycle plays an important rôle.

The following theorem is a generalization of Theorem 2.6.

Theorem 2.8. All continuous functions permutable with the continuous function $k(x)\ m(y)$ are of the form $k(x)\cdot m(y)\cdot G(m^{*}k)$.

Proof: The problem is to find the most general continuous function, $g(x,y)$, that satisfies the equation

$$\int_x^y k(x)\ m(t)\ g(t,y)dt = \int_x^y g(x,t)\ k(t)\ m(y)dt.$$

If one sets $g(x,y) = k(x)\ m(y)\ h(x,y)$, and divides by $k(x)\ m(y)$, the above equation becomes

$$\int_x^y m(t)\ k(t)\ h(t,y)dt = \int_x^y m(t)\ k(t)\ h(x,t)dt. \qquad (2.6)$$

Now choose an interval in which $k(t)\ m(t)$ does not change sign. In this interval $k(t)\ m(t)$ may be assumed to be greater than zero. If one sets

$$u = \int_0^t m(w)\ k(w)dw = q(t),$$

then

$$du = m(t)\ k(t)dt.$$

Since u is a monotonically increasing function of t, a unique inverse function $t = q^{-1}(u)$ exists. If one makes the change of variable from t to u in equation (2.6), the following equation is obtained

$$\int_{q(x)}^{q(y)} h\left[q^{-1}(u),y\right]du = \int_{q(x)}^{q(y)} h\left[x,q^{-1}(u)\right]du.$$

If one sets $y = q^{-1}(z)$ and $x = q^{-1}(w)$, the above equation becomes

$$\int_{w}^{z} h\left[q^{-1}(u),\ q^{-1}(z)\right]du = \int_{w}^{z} h\left[q^{-1}(w),\ q^{-1}(u)\right]du.$$

If $Q(w,z)$ is set equal to $h\left[q^{-1}(w),\ q^{-1}(z)\right]$, the above equation reduces to the following equation

$$\int_{w}^{z} Q(u,z)du = \int_{w}^{z} Q(w,u)du.$$

But by Theorem 2.6 the only functions which satisfy this equation are functions of the difference z-w, say $G(z-w)$. Thus

$$Q(w,z) = G(z-w) = G\left[q(y) - q(x)\right] = h(x,y).$$

Hence

$$h(x,y) = G\left[\int_{0}^{y} m(w)\ k(w)dw - \int_{0}^{x} m(w)\ k(w)dw\right]$$

$$= G\left[\int_{x}^{y} m(w)\ k(w)dw\right]$$

$$= G(m*k).$$

Since this condition is also sufficient, and since

$$g(x,y) = k(x)\ m(y)\ h(x,y) = k(x)\ m(y)\ G(m^*k),$$

the theorem is proved.

As a particular example of this theorem it is easily seen that all continuous functions permutable with $x^p y^q$ are of the form $x^p y^q \phi\left[y^{p+q-1} - x^{p+q+1}\right]$ for in this case $k(x) = x^p$, $m(y) = y^q$ and $m^*k = \dfrac{y^{p+q+1} - x^{p+q+1}}{p+q+1}$.

<u>Definition</u> 2.5. If the function $f(x,y)$ is equal to

$$\frac{(y-x)^{p-1}}{\Gamma(p)} G(x,y),\ p > 0,$$

where the function $G(x,y)$ is finite and continuous and $G(x,x) \neq 0$, then $f(x,y)$ is said to be of <u>order</u> p.

The following theorem is an important result in the theory of permutable functions (18, p. 52).

<u>Theorem</u> 2.9. <u>All analytic functions</u> $g(x,y)$ <u>which are permutable with</u> $f(x,y)$ <u>of order one are of the form</u> $g(x,y) = \sum_{i=1}^{\infty} a_i f^{*i}(x,y)$, <u>where the</u> a_i <u>are subject to the restriction that the power series</u> $\sum_{i=1}^{\infty} a_i \frac{z^i}{i!}$ <u>converge</u>.

Since the function $(x+y)$ is of order one, it follows that all functions permutable with $(x+y)$ are of the form

$$g(x,y) = \sum_{i=1}^{\infty} a_i (x+y)^{*i},$$

where the a_i are subject to the restriction of Theorem 2.9. Use is made of this result in the proof of the following theorem.

<u>Theorem</u> 2.10. <u>The following matrix which has</u> (n+3) <u>rows</u>

and (n+1) columns has rank (n+1) if n is even and rank less than (n+1) if n is odd .

$$
\begin{bmatrix}
\frac{1}{n+2} & \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{4} & \frac{1}{3} & \frac{1}{2}-\frac{1}{n+2}-\frac{1}{n+1} \\
\frac{1}{n+1} & \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{3} & \frac{1}{2}-\frac{1}{n+1}-\frac{1}{n} & 1 \\
0 & 0 & 0 & \cdots & -\frac{1}{n}-\frac{1}{n-1} & 0 & -1-\frac{1}{2} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{2}-\frac{1}{3} & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
-\frac{1}{2}-1 & 0 & -\frac{1}{n}-\frac{1}{n-1} & \cdots & 0 & 0 & 0 \\
1 & \frac{1}{2}-\frac{1}{n+1}-\frac{1}{n} & \frac{1}{3} & \cdots & \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} \\
\frac{1}{2}-\frac{1}{n+2}-\frac{1}{n+1} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2}
\end{bmatrix}
$$

Proof: Assume $g(x,y)$ is a homogeneous polynomial of degree n, that is, $g(x,y) = a_o x^n + a_1 y\, x^{n-1} + \ldots + a_{n-1} y^{n-1} x + a_n y^n$. If $g(x,y)$ is to be permutable with $(x+y)$, the following equation must be satisfied:

$$\int_x^y (x+t)(a_o t^n + a_1 y t^{n-1} + \ldots + a_{n-1} y^{n-1}\, t + a_n y^n)\,dt$$

$$-\int_x^y (t+y)(a_o x^n + a_1 x^{n-1} t + \ldots + a_{n-1} t^{n-1} x + a_n t^n)\,dt = 0.$$

In order that the equation which is obtained after integration may be satisfied, the coefficient of $x^i y^j$ for $0 \leq i,\ j \leq (n+2)$ must be zero. These $(n+3)$ coefficients which must be zero are the rows in the above matrix. Thus one obtains $(n+3)$ homogeneous linear equations in the $(n+1)$ unknowns: $a_0, a_1, \ldots, a_n$. These equations have a nontrivial solution if, and only if, the rank of the matrix of the coefficients is less than $(n+1)$. The function $(x+y)^{*n}$ is a homogeneous polynomial of degree $(2n-1)$. This follows directly from the method of construction of these powers as given in section C of this chapter. Hence by Theorem 2.9 the degree of $g(x,y)$ cannot be even, but for each odd n, $= 2k-1$, there exists a polynomial $g(x,y) = (x+y)^{*k}$ which satisfies the above equation. Hence if $g(x,y)$ is of even degree, there exists no nontrivial solution for the system of equations and thus the rank of the matrix must be equal to $(n+1)$; but if $g(x,y)$ is of odd degree, there exists a solution and thus the rank is less than $(n+1)$. This proves the theorem.

The following question now arises. Can all functions permutable with a given function $f(x,y)$ of any positive integral order be expressed as a power series of composition in terms of the given function as in Theorem 2.9? The answer is clearly no, since 1 and $(y-x)$ are permutable but for any choice of the a_i, $\sum_{i=1}^{\infty} a_i(y-x)^{*n} \neq 1$. Peres (9) has considered this more general problem of finding all analytic functions permutable with a given function of positive integral order.

E. Another Type of Composition

Suppose the product by composition of $f(x,y)$ and $g(x,y)$ is defined with Davis (3) as

$$\int_{A(x)}^{B(y)} f(x,t)\, g(t,y)dt\,,$$

where $A(x)$ and $B(y)$ are analytic functions and $f(x,y)$ and $g(x,y)$ are restricted in so far that the above integral exists. This is, of course, a generalization of the previous type of composition and reduces to that, if $A(x) = x$ and $B(y) = y$. Immediately one asks the following question: For what $A(x)$ and $B(y)$ will the set of postulates given by Evans in section B of this chapter be satisfied? The following theorem answers this question.

Theorem 2.11. The associative law $f^*(g^*h) = (f^*g)^*h$ will be satisfied if and only if $A(x) = x$ and $B(y) = y$ or if $A(x) = a$ and $B(y) = b$, where a and b are arbitrary constants. (The asterisk will also denote this type of composition.)

Proof: It is easily shown that

$$f^*(g^*h) = \int_{A(x)}^{B(y)} \int_{A(t)}^{B(y)} f(x,t)\, g(t,s)\, h(s,y)dsdt$$

and that

$$(f^*g)^*h = \int_{A(x)}^{B(y)} \int_{A(x)}^{B(s)} f(x,t)\, g(t,s)\, h(s,y)dtds.$$

Hence, if the associative law is satisfied, then

$$\int_{A(x)}^{B(y)} \int_{A(t)}^{B(y)} f(x,t)\, g(t,s)\, h(s,y) ds dt$$

$$= \int_{A(x)}^{B(y)} \int_{A(x)}^{B(s)} f(x,t)\, g(t,s)\, h(s,y) dt ds. \qquad (2.7)$$

This must be true for all continuous f,g, and h, at least. In particular, it must be true if $f = g = h = 1$. The above equation then reduces to the following

$$\int_{A(x)}^{B(y)} \int_{A(t)}^{B(y)} ds dt = \int_{A(x)}^{B(y)} \int_{A(x)}^{B(s)} dt ds.$$

Carrying out the indicated integration, one can easily show that $A(x)$ and $B(y)$ must satisfy the following equation

$$B^2(y) - \int_{A(x)}^{B(y)} A(t) dt = A^2(x) + \int_{A(x)}^{B(y)} B(s) ds.$$

Since this equation is to be an identity in x and y, the partial derivative of both sides of this equation with respect to x must be equal, as must also the partial derivative of both sides with respect to y. Taking the partial derivative of both sides of the equation with respect to x, and with respect to y, one obtains the following equations

$$A'(x) \left\{ 2A(x) - B[A(x)] - A[A(x)] \right\} = 0$$
$$B'(y) \left\{ 2B(y) - A[B(y)] - B[B(y)] \right\} = 0.$$

Hence, at least one of the following necessary conditions must be satisfied

$$\begin{cases} A'(u) \equiv 0 \\ B'(u) \equiv 0 \end{cases}$$

$$\begin{cases} A'(u) \equiv 0 \\ A(u)+B(u) \equiv 2u \end{cases}$$

$$\begin{cases} B'(u) \equiv 0 \\ A(u)+B(u) \equiv 2u \end{cases}$$

$$\left[A(u)+B(u) \equiv 2u \right. .$$

The first set of conditions implies that $A(u) = a$ and $B(u) = b$; the second set that $A(u) = a_2$, $B(u) = 2u-a_2$; the third set that $B(u) = a_3$ and $A(u) = 2u-a_3$. In order to prove the theorem it is convenient to obtain an additional set of necessary conditions. To this end, set $f(x,y) = y$, $g=h=1$, in equation (2.7). Taking the partial derivative of both sides of this equation with respect to y, one obtains the following necessary conditions

$$\begin{aligned} B'(u) &\equiv 0 \\ B^2(u)+2u\,A(u)-3u^2 &\equiv 0 \ . \end{aligned} \tag{2.8}$$

The second and third set of solutions above do not satisfy this set of necessary conditions and hence are inadmissible. By substituting $A(u) = 2u-B(u)$ in equation (2.8), one can easily show that $A(x) = x$ and $B(y) = y$.

The conditions $A(x) = x$ and $B(y) = y$ are sufficient by Theorem 2.1. It can also be easily shown that $A(x) = a$ and $B(x) = b$ is another sufficient condition. This is the composition of the second kind which was mentioned in the beginning of this chapter. The theorem is thus proved.

Since the associative law is not satisfied unless $A(x) = x$ and $B(y) = y$ or $A(x) = a$ and $B(y) = b$, the set of postulates given by Evans will not be satisfied. Hence, the only types of composition which satisfy Evans' postulates are Volterra's compositions of the first and second kind.

III. THE VOLTERRA TRANSFORM

A. The Hypergeometric Series

The series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!} $$

where $(a)_n = a(a+1)(a+2)\dots(a+n-1)$, $(a)_0 = 1$, is called Gauss's hypergeometric series. More generally the series

$$ {}_pF_q\left[a_1,\dots,a_p;b_1,\dots,b_q;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n z^n}{n!(b_1)_n\cdots(b_q)_n} \qquad (3.1) $$

is called the generalized hypergeometric series. The following theorem can be proved concerning the generalized hypergeometric series (2).

<u>Theorem</u> 3.1. <u>When</u> $p \leq q$, <u>the series converges for all values of</u> z. <u>When</u> $p > q+1$, <u>the series converges only for</u> $z = 0$ <u>unless it terminates and it is therefore significant only when it terminates</u>. <u>For</u> $p = q+1$, <u>the series converges when</u> $|z| < 1$, <u>and also when</u> $z = 1$ <u>provided that the real part of</u> $(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i) > 0$, <u>and when</u> $z = -1$ <u>provided that the real part of</u> $(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i + 1) > 0$.

The generalized hypergeometric series is a solution of the following differential equation (14).

$$ \left[(z\tfrac{d}{dz}+a_1)\dots(z\tfrac{d}{dz}+a_p) - \tfrac{d}{dz}(\tfrac{zd}{dz}+b_1-1)\dots(z\tfrac{d}{dz}+b_q-1)\right] y = 0. $$

In particular, if $p = 2$ and $q = 1$ the generalized hypergeometric series reduces to Gauss's hypergeometric series and the above differential equation reduces to

$$z(z-1)\,\frac{d^2y}{dz^2} - \Big[b_1-(a_1+a_2+1)z\Big]\frac{dy}{dz}+a_1a_2y = 0. \qquad (3.2)$$

B. Definition of the Volterra Transform

<u>Definition</u> 3.1. The <u>Volterra</u> <u>transform</u> of a polynomial or an infinite series in z is the function obtained from the given function by replacing powers of z by the corresponding powers by composition of $f(x,y) = 1$. The transform of a $g(z)$ will be denoted by $V\big[g(z)\big]$.

As an illustration, the Volterra transform of

$$z^p\ {}_2F_1(a,b;c;z),\ p>0,$$

is given here.

$$V\Big[z^p\ {}_2F_1(a,b;c;z)\Big] = V\Big[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^{n+p}}{n!}\Big]$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{1^{*n+p}}{n!},$$

and since $1^{*a} = \dfrac{(y-x)^{a-1}}{\Gamma(a)}$,

$$V\left[z^p\ {}_2F_1(a,b;c;z)\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(y-x)^{n+p-1}}{(c)_n\Gamma(n+p)\ n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(y-x)^{n+p-1}}{(c)_n(p)_n\ \Gamma(p)n!}$$

$$= \frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_2F_2(a,b;c,p;y-x).$$

For easy reference the Volterra transforms of the particular hypergeometric series which are considered throughout the thesis are tabulated in Table 1 (see next page).

C. Some Applications of the Volterra Transform

1. Evaluation of an integral

The Volterra transform enables one to derive many important properties of some hypergeometric series. Some examples of the properties obtainable by this method are given by the theorems in this section. The following theorem illustrates how equations involving products of hypergeometric series can be used to evaluate certain integrals of hypergeometric series.

Theorem 3.2. The hypergeometric series ${}_2F_1$ and ${}_4F_3$ satisfy the following equation

$$\int_x^y (t-x)^{p-1}\ {}_2F_1(a,b;p;t-x)(y-t)^{p-1}\ {}_2F_1(a,b;p;t-y)dt$$

$$= \frac{\Gamma^2(p)}{\Gamma(2p)}(y-x)^{2p-1}\ {}_4F_3\left[a,b,\frac{a+b}{2},\frac{a+b+1}{2};a+b,p,p+\frac{1}{2};(y-x)^2\right]$$

where $p>0$, a or b a negative integer.

Table 1

Volterra Transforms of Some Hypergeometric Series

	Function	Transform
1.	$z^p {}_2F_1(a,b;c;z)$, $p>0$	$\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_2F_2(a,b;c,p;y-x)$
2.	$z^p {}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;z)$, $p>0$	$\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_rF_{s+1}(a_1,\ldots,a_r;b_1,\ldots,b_s,p;y-x)$
3.	$i^{n+1}\ O_n(\frac{i}{z})$, $i^2=-1$	$T_n(y-x)$
4.	$\frac{z^p}{(1-z)^p}$, $p>0$	$\frac{e^{y-x}(y-x)^{p-1}}{\Gamma(p)}$
5.	$\frac{z^a}{(1-z)^a}\ {}_2F_1(a,b;c;\frac{z}{1-z})$, $a>0$	$\frac{e^{y-x}(y-x)^{a-1}}{\Gamma(a)}\ {}_1F_1(b;c;y-x)$
6.	$J_p(2\sqrt{z})$, $p>0$	$\frac{(y-x)^{p/2-1}}{\Gamma(p+1)\Gamma(p/2)}\ {}_0F_2(p+1,\frac{p}{2};x-y)$
7.	$z^{2p} {}_4F_1(a,b,\frac{a+b}{2},\frac{a+b+1}{2};a+b;4z^2)$, $p>0$	$\frac{(y-x)^{2p-1}}{\Gamma(2p)}\ {}_4F_3\left[a,b,\frac{a+b}{2},\frac{a+b+1}{2};a+b,p,p+\frac{1}{2};(y-x)^2\right]$
8.	$z^{2a} {}_2F_3(a,p-a;p,\frac{p}{2},\frac{p+1}{2};\frac{z^2}{4})$, $a>0$	$\frac{(y-x)^{2a-1}}{\Gamma(2a)}\ {}_1F_4\left[p-a;\frac{1}{2}+a,p,\frac{p}{2},\frac{p+1}{2};\frac{(y-x)^2}{2^4}\right]$
9.	$z^{v+p}\ e^{hz}\ {}_0F_1(v+c;-z)$, $v+p>0$	$\sum_{i=0}^{\infty}\frac{(y-x)^{v+p+i-1}(-1)^i\Gamma(v+c)\,{}_0F_1\left[v+p+i,h(y-x)\right]}{\Gamma(v+p+i)\ \Gamma(v+c+i)\ i!}$
10.	$z^p\ T_n(1-2z)$, $p>0$	$\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_2F_2(n,-n;\frac{1}{2},p;y-x)$
11.	$e^{-az}\ z^u$, $u>0$	$\left(\frac{y-x}{a}\right)^{\frac{u-1}{2}}\ J_{u-1}\left(2\sqrt{a(y-x)}\right)$

Proof: Bailey (1) has shown that

$$ {}_2F_0(a,b;z)\ {}_2F_0(a,b;-z) = {}_4F_1(a,b,\ \frac{a+b}{2},\ \frac{a+b+1}{2};\ a+b;\ 4z^2). $$

If both sides of this equation are multiplied by z^{2p}, $p > 0$, the following equation is obtained.

$$ \left[z^p\,{}_2F_0(a,b;z)\right]\cdot\left[z^p\,{}_2F_0(a,b;-z)\right] = z^{2p}\,{}_4F_1\left[a,b,\frac{a+b}{2},\frac{a+b+1}{2};a+b;4z^2\right]. $$

Taking the Volterra transform of both sides with the terms arranged as indicated one obtains the following equation

$$ V\left[z^p\,{}_2F_0(a,b;z)\right] {}^{*}V\left[z^p\,{}_2F_0(a,b;-z)\right] = V\left[z^{2p}\,{}_4F_1(a,b,\frac{a+b}{2},\frac{a+b+1}{2};a+b;4z^2)\right]. $$

By making use of the table of transforms and rewriting the above equation, one obtains the desired result. Even though the integrands above are not continuous, it is clear that the integrals will always exist.

2. Recursion formulas

The following theorem shows how recursion formulas for certain hypergeometric series can be obtained from known recursion formulas of hypergeometric series.

Theorem 3.3. The hypergeometric series ${}_2F_2$ satisfy the following recursion formula

$$ \begin{aligned} cp\ {}_2F_2(a,b-1;c,p;t) - cp\ {}_2F_2(a-1,b;c,p;t) \\ + (a-b)t\ {}_2F_2(a,b;c+1,p+1;t) = 0 \end{aligned} $$

where $p > 0$.

Proof: One of Gauss's well known recursion formulas (8, p. 9) for the hypergeometric series ${}_2F_1$ is

$$c\,{}_2F_1(a,b-1;c;z) - c\,{}_2F_1(a-1,b;c;z) + (a-b)z\,{}_2F_1(a,b;c+1;z) = 0.$$

If each term of this equation is multiplied by z^p, $p>0$, then

$$cz^p\,{}_2F_1(a,b-1;c;z) - cz^p\,{}_2F_1(a-1,b;c;z)+(a-b)z^{p+1}\,{}_2F_1(a,b;c+1;z)=0.$$

By taking the Volterra transform of each term of this equation, one obtains the following equation

$$c\frac{(y-x)^{p-1}}{\Gamma(p)}\,{}_2F_2(a,b-1;c,p;y-x)-c\frac{(y-x)^{p-1}}{\Gamma(p)}\,{}_2F_2(a-1,b;c,p;y-x)$$

$$+\ (a-b)\frac{(y-x)^p}{\Gamma(p+1)}\,{}_2F_2(a,b;c+1,p+1;y-x) = 0.$$

If $(y-x)$ is set equal to t, and each term of the above equation is divided by $\frac{t^{p-1}}{\Gamma(p)}$, the desired recursion formula is obtained.

It is clear that by a similar procedure a recursion formula for ${}_2F_3$ can be obtained from that given for ${}_2F_2$ and in general in a similar manner a recursion formula for ${}_2F_n$ can be derived.

3. Series expansions

The following two theorems show how new series expansions can be obtained by taking the Volterra transform of both sides of known series expansions of certain functions.

Theorem 3.4. Bateman's polynomial $J_n^{u,w}(t)$ can be expanded in a double sum of Bessel functions in the following manner

$$J_n^{u,w}(t)$$

$$= \frac{\Gamma(w+\frac{u}{2}+n+1)}{n!} \sum_{v=0}^{\infty} \sum_{i=0}^{\infty} \frac{A_v(-1)^i\, t^{(-v-w-\frac{u}{2}+i)/2}\, n^{v+i-w+\frac{u}{2}}}{\Gamma(v+u+i+1)\, i!\, h^{(v+w+\frac{u}{2}+i)/2}} \cdot J_{v+w+\frac{u}{2}+i}(2t\sqrt{-h}),$$

where $J_n^{u,w}(t)$ is defined as (25)

$${}_1F_2(-n;u+1,\ w+\tfrac{u}{2}+1;\ t^2) = \frac{t^{-u} J_n^{u,w}(t)\ n!\,\Gamma(u+1)\,\Gamma(w+\frac{u}{2}+1)}{\Gamma(w+\frac{u}{2}+n+1)},$$

where

$$e^{ax} \frac{[1+(h-1)x]^a}{[1+hx]^{a+c}} = \sum_{v=0}^{\infty} A_v\, x^v,$$

and where

$$w + \frac{u}{2} + 1 > 0,\quad c = u+1,\quad a = n.$$

Proof: If in the known identity (13)

$${}_1F_1(-a;c;\tfrac{t}{a}) = \Gamma(c)\ t^{\frac{1-c}{2}}\ e^{\frac{ht}{a}} \sum_{v=0}^{\infty} \frac{A_v}{a^v}\, t^{v/2}\, J_{v+c-1}(2\sqrt{t}),$$

$\frac{t}{a}$ is replaced by z, the following equation is obtained

$${}_1F_1(-a;c;z) = \Gamma(c)(az)^{\frac{1-c}{2}}\ e^{hz} \sum_{v=0}^{\infty} \frac{A_v}{a^{v/2}}\, z^{v/2}\, J_{v+c-1}(2\sqrt{az}). \tag{3.3}$$

Multiplying this equation by z^p, $p>0$, and taking the Volterra transform of both sides, one obtains the following equation

$$\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_1F_2(-a;c,p;y-x) = \Gamma(c)\ a^{\frac{1-c}{2}} \sum_{v=0}^{\infty} \frac{A_v}{a^{v/2}}$$

$$\cdot V\left[z^{\frac{1-c+v}{2}+p}\ e^{hz}\ J_{v+c-1}(2\sqrt{az})\right].$$

Since

$$J_{v+c-1}(2\sqrt{az}) = \frac{(az)^{\frac{v+c-1}{2}}}{\Gamma(v+c)}\ {}_0F_1(v+c;-az),$$

the above equation may be written with the use of the table of transforms as

$$\frac{(y-x)^{p-1}}{\Gamma(p)\Gamma(c)}\ {}_1F_2(-a;c,p;y-x)$$

$$= \sum_{v=0}^{\infty}\sum_{i=0}^{\infty} \frac{A_v (y-x)^{p+v+i-1}(-1)^i\ a^i\ {}_0F_1[v+p+i,\ h(y-x)]}{\Gamma(v+p+i)\ \ \Gamma(v+c+i)\ i\,!}\ . \qquad (3.4)$$

By writing this equation in terms of Bessel functions one obtains the following equation

$$\frac{{}_1F_2(-a;c,p;y-x)}{\Gamma(p)\Gamma(c)}$$

$$= \sum_{v=0}^{\infty}\sum_{i=0}^{\infty} \frac{A_v\ (-1)^{\frac{-v-p+i+1}{2}}\ (y-x)^{\frac{v+i-p+1}{2}}\ a^i\ J_{v+p+i-1}\left[2\sqrt{h(x-y)}\right]}{\Gamma(v+c+i)\ i\,!\ h^{\frac{v+i-1+p}{2}}}\ .$$

If one sets $a = n$, $c = u+1$, $p = w+\frac{u}{2}+1$, $y-x = t^2$, the above equation becomes

$$\frac{{}_1F_2(-n;u+1,w+\frac{u}{2}+1;t^2)}{\Gamma(w+\frac{u}{2}+1)\Gamma(u+1)} = \sum_{v=0}^{\infty}\sum_{i=0}^{\infty} \frac{A_v(-1)^{(-v-w-\frac{u}{2}+i)/2}\ t^{v+i-w-\frac{u}{2}}\ n^i}{\Gamma(v+u+1+i)\ i\,!\ h^{(v+w+\frac{u}{2}+i)/2}}$$

$$\cdot\ J_{v+w+\frac{u}{2}+i}\ (2t\sqrt{-h}).$$

The desired result is then obtained if $J_n^{u,w}(t)$ is substituted for ${}_1F_2(-n;u+1,w+\frac{u}{2}+1;t^2)$ in the above equation. From (3.4) it is clear that if $(y-x)$ is replaced by z and the Volterra transform of this equation is taken, an expansion of hypergeometric series of the type ${}_1F_3$ is obtained in terms of a double sum of hypergeometric series of the type ${}_1F_1$. In general then by a similar procedure ${}_1F_n$ can be expanded in terms of a double sum of hypergeometric series of the type ${}_1F_{n-2}$.

Theorem 3.5. The hypergeometric series ${}_oF_{2p-1}$ satisfy the following equation

$$1 = \sum_{n=0}^{\infty} \frac{\epsilon_n}{\left[\Gamma(2n+1)\right]^p} \left(\frac{z}{2}\right)^{2n} {}_oF_{2p-1}\Big[2n+1, \overbrace{n+1, \ldots, n+1}^{p-1}, \overbrace{n+\tfrac{1}{2}, \ldots, n+\tfrac{1}{2}}^{p-1}; -\frac{z^2}{4^p}\Big]$$

where $\epsilon_n = 2$ for $n \geq 1$, $\epsilon_o = 1$, $p = 1,2,\ldots$.

Proof: One can obtain the desired result by successively multiplying the known equation (8, p. 19)

$$1 = \sum_{n=0}^{\infty} \epsilon_n J_{2n}(z)$$

by z and by taking the Volterra transform of the terms of each resulting equation. For example, after multiplying the above equation by z and taking the Volterra transform of each term, one obtains the following equation ($z = y-x$)

$$1 = \sum_{n=0}^{\infty} \frac{\epsilon_n z^{2n}}{2^{2n}\left[\Gamma(2n+1)\right]^2} {}_oF_3\Big[2n+1, n+1, n+\tfrac{1}{2}; \frac{-z^2}{4^2}\Big].$$

Multiplying this equation by z and then taking the Volterra transform one obtains the following equation

$$1 = \sum_{n=0}^{\infty} \frac{\epsilon_n z^{2n}}{2^{2n}\left[\Gamma(2n+1)\right]^3} {}_oF_5\Big[2n+1, n+1, n+1, n+\tfrac{1}{2}, n+\tfrac{1}{2}; \frac{-z^2}{4^3}\Big].$$

By the use of mathematical induction, it is clear that one can easily obtain the desired result.

4. Use of the inverse Volterra transform

Another method of obtaining new series expansions of certain functions is to consider a known series expansion of a function as the Volterra transform of some other series expansion of a function, if this is possible. Then this other series is said to be obtainable by the inverse Volterra transform of the given series. The following theorem is an illustration of the use of this method.

Theorem 3.6. The Gaussian hypergeometric series ${}_2F_1(b,-a;c;\frac{z}{1-z})$ can be expanded in terms of the confluent hypergeometric series ${}_1F_1(v+b;v+c;\frac{-az}{1-(h+1)z})$ in the following manner

$$\frac{z^b}{(1-z)^b}\,{}_2F_1(b,-a;c;\frac{z}{1-z})$$

$$= \frac{\Gamma(c)}{\Gamma(b)}\sum_{v=0}^{\infty}\frac{A_v\, z^{v+b}\Gamma(v+b)\;{}_1F_1(v+b;v+c;\frac{-az}{1-(h+1)z})}{\Gamma(v+c)\left[1-(h+1)z\right]^{v+b}}$$

where $b>0$.

Proof: The following identity can be obtained if one multiplies equation (3.3) by $e^{y-x}\frac{(y-x)^{b-1}}{\Gamma(b)}$, $b>0$, and replaces z by y-x,

$$e^{(y-x)}\,\frac{(y-x)^{b-1}}{\Gamma(b)}\,{}_1F_1(-a;c;y-x)$$

$$= e^{y-x}\,\frac{(y-x)^{b-1}}{\Gamma(b)}\Gamma(c)\left[a(y-x)\right]^{\frac{1-c}{2}}e^{h(y-x)}\sum_{v=0}^{\infty}\frac{A_v}{a^{\frac{v}{2}}}(y-x)^{\frac{v}{2}}J_{v+c-1}\left[2\sqrt{a(y-x)}\right].$$

If the terms of this equation are considered as the Volterra transforms of the terms of some original equation, then the desired result is obtained by the use of Table 1. It is thus seen that the equation in Theorem 3.6 was obtained from the last equation by taking the inverse Volterra transform of the terms of this latter equation.

IV. TCHEBYCHEFF POLYNOMIALS

A. Definitions and Theorems

In this chapter some identities concerning Tchebycheff polynomials will be derived, some of which prove to be useful in the next chapter.

__Definition__ 4.1. The __Tchebycheff__ __polynomials__ are defined by

$$T_n(x) = \cos(n \text{ arc } \cos x). \quad (n = 0, 1, 2, \ldots)$$

It can easily be verified that these polynomials satisfy the following differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} + n^2y = 0.$$

Since this differential equation can be obtained from equation (3.2) if one sets $z = \frac{1-x}{2}$, $b_1 = \frac{1}{2}$, $a_1 = -a_2 = n$, it follows that

$$T_n(x) = {}_2F_1(n,-n;\tfrac{1}{2};\tfrac{1-x}{2}) .$$

It is well known that the Tchebycheff polynomials satisfy the recursion relation or algebraic addition formula

$$T_{n+m}(x) + T_{n-m}(x) = 2T_n(x)\, T_m(x) . \tag{4.1}$$

Expressing this equation in terms of hypergeometric series, one obtains the following algebraic addition formula

$$2\ {}_2F_1(n,-n;\tfrac{1}{2};\tfrac{1-x}{2})\,{}_2F_1(m,-m;\tfrac{1}{2};\tfrac{1-x}{2})$$

$$= {}_2F_1(n-m,m-n;\tfrac{1}{2};\tfrac{1-x}{2}) + {}_2F_1(n+m,-n-m;\tfrac{1}{2};\tfrac{1-x}{2}). \qquad (4.2)$$

Theorem 4.1. *The Tchebycheff polynomials satisfy the following equation*

$$\sum_{i=2}^{p} 2^{i-2}T_m^{\,i-2}(x)T_{im}(x) = 2^{p-1}T_m^{\,p-1}(x)T_{(p-1)m}(x) - T_o(x).$$

Proof: If one sets successively $n=m$, $2m$, $3m,\ldots,$ $(p-1)m$ in equation (4.1) the following set of equations results

$$\begin{aligned}
T_{2m}(x) &= 2T_m^{\,2}(x) - T_o(x)\\
T_{3m}(x) &= 2T_{2m}(x)\ T_m(x) - T_m(x)\\
T_{4m}(x) &= 2T_{3m}(x)\ T_m(x) - T_{2m}(x)\\
&\ \ \vdots\\
T_{pm}(x) &= 2T_{(p-1)m}(x)\ T_m(x) - T_{(p-2)m}(x).
\end{aligned}$$

Multiplying the second equation by $2T_m(x)$, the third equation by $2^2T_m^{\,2}(x)$, and so on, the $(p-1)$ st equation by $2^{p-2}T_m^{\,p-2}(x)$, one obtains the following set of equations

$$\begin{aligned}
T_{2m}(x) &= 2T_m^{\,2}(x) - T_o(x)\\
2T_m(x)\ T_{3m}(x) &= 2^2T_{2m}(x)\ T_m^{\,2}(x) - 2T_m^{\,2}(x)\\
2^2T_m^{\,2}(x)\ T_{4m}(x) &= 2^3T_{3m}(x)\ T_m^{\,3}(x) - 2^2T_m^{\,2}(x)\ T_{2m}(x)\\
&\ \ \vdots\\
2^{p-2}T_m^{\,p-2}(x)T_{pm}(x) &= 2^{p-1}T_m^{\,p-1}(x)T_{(p-1)m}(x) - 2^{p-2}T_{(p-2)m}(x)T_m^{\,p-2}(x).
\end{aligned}$$

By adding the equations of this set one obtains the desired result.

B. Proofs of Some Theorems By the Use of Commutativity

<u>Definition</u> 4.2. The <u>functional product</u> of two functions $f(x)$ and $g(x)$ has been defined (20) as $f[g(x)]$ and designated by fg. Two functions are said to commute with each other if $fg = gf$.

It has been shown (11) that the Tchebycheff polynomials are an entire set of commutative polynomials, that is, a set of polynomials which contains at least one of each positive degree, and which is such that any two polynomials of the set commute with each other. Thus

$$T_n[T_m(x)] = T_m[T_n(x)] = T_{nm}(x).$$

Hence, if one substitutes $T_p(x)$ for x in equation (4.1) the following algebraic addition formula is obtained

$$T_{(n+m)p}(x) + T_{(n-m)p}(x) = 2T_{np}(x)\, T_{mp}(x).$$

This equation could be generalized further if one replaces x by $T_q(x)$ in the above equation. If one replaces x by $T_q(x)$ in the equation of Theorem 4.1 the following equation is obtained

$$\sum_{i=2}^{p} 2^{i-2}\, T_{mq}^{\,i-2}(x)\, T_{imq}(x) = 2^{p-1}\, T_{mq}^{\,p-1}(x)\, T_{(p-1)mq}(x) - T_o(x),$$

which could be generalized again by the use of commutativity.

The Tchebycheff polynomials are orthogonal on the interval $-1 \leq x \leq 1$ with weight function $(1-x^2)^{-1/2}$, that is,

$$\int_{-1}^{1} \frac{T_m(x)\ T_n(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\delta_{mn} \qquad m,\ n \geq 1$$

where δ_{mn} is the Kronecker delta which is equal to zero for $m \neq n$ and equal to one for $m = n$.

Some interesting identities involving Tchebycheff polynomials can be obtained if one expands a function $f(x)$ in Tchebycheff polynomials and then uses the commutative property of these polynomials. The following theorem is an example.

<u>Theorem</u> 4.2 <u>For</u> $|x| < 1$,

$$\left[\sum_{m=0}^{\infty} \frac{\epsilon_m}{\pi(1+m^2)}(e^{\pi}\cos m\pi-1)T_m(x)\right]^p = \sum_{m=0}^{\infty} \frac{\epsilon_m}{\pi(1+m^2)}(e^{\pi}\cos m\pi-1)T_{mp}(x).$$

<u>Proof</u>: If $f(x) = \sum_{m=0}^{\infty} a_m\ T_m(x)$,

then

$$a_m = \frac{\epsilon_m}{\pi}\int_{-1}^{1} \frac{f(x)T_m(x)}{\sqrt{1-x^2}}\ dx.$$

If one sets $f(x) = e^{\mathrm{Arc}\cos x}$, where $\mathrm{Arc}\cos x$ stands for the principal value of this function, then

$$a_m = \frac{\epsilon_m}{\pi}\int_{-1}^{1} e^{\mathrm{Arc}\cos x}\ \frac{\cos m \operatorname{arc}\cos x}{\sqrt{1-x^2}}\ dx.$$

By the substitution of $u = \mathrm{Arc}\cos x$ in the above integral, one can easily show that

$$a_m = \frac{\epsilon_m}{\pi(1+m^2)} (e^{\pi} \cos m\pi - 1).$$

Thus

$$e^{\text{Arc cos } x} = \sum_{m=0}^{\infty} \frac{\epsilon_m}{\pi(1+m^2)} (e^{\pi} \cos m\pi - 1) T_m(x). \qquad (4.3)$$

Raising both sides of this equation to the pth power, one obtains the following result

$$e^{p \text{ Arc cos } x} = \left[\sum_{m=0}^{\infty} \frac{\epsilon_m}{\pi(1+m^2)} (e^{\pi} \cos m\pi - 1) T_m(x)\right]^p.$$

However, replacing x by $T_p(x)$ in equation (4.3), one obtains another expression for $e^{p \text{ Arc cos } x}$ which, if equated to the right-hand side of the above equation, gives the desired result.

<u>Theorem</u> 4.3. <u>The uth power of the Tchebycheff polynomial can be expressed as a sum of Tchebycheff polynomials in the following manner</u>

$$2^u \, T_p{}^u(x) = \sum_{k=0}^{[\frac{u}{2}]} \epsilon_{[\frac{u+1}{2}]-k} \binom{u}{k} T_{(u-2k)p}(x),$$

<u>where</u> $[\frac{u}{2}]$ <u>denotes the largest integer in</u> $\frac{u}{2}$.

<u>Proof:</u> It is known (7) that

$$2^u \, x^u = \sum_{k=0}^{[\frac{u}{2}]} \epsilon_{[\frac{u+1}{2}]-k} \binom{u}{k} T_{u-2k}(x).$$

Replacing x by $T_p(x)$ in the above equation, one obtains the desired result.

Theorem 4.4. The Tchebycheff polynomials satisfy the following well known Christoffel relation (10).

$$1 + 2\sum_{v=1}^{n} T_v(x)\ T_v(y) = \frac{T_{n+1}(x)\ T_n(y) - T_n(x)\ T_{n+1}(y)}{x-y}.$$

Proof: The setting of $m = 1$ and $n = v$ in equation (4.1) yields

$$2x\ T_v(x) = T_{v+1}(x) + T_{v-1}(x),$$

which, if $x = y$, becomes

$$2y\ T_v(y) = T_{v+1}(y) + T_{v-1}(y).$$

The multiplication of the first displayed equation by $T_v(y)$ and the second one by $T_v(x)$ results in the following equations

$$2x\ T_v(x)\ T_v(y) = T_{v+1}(x)\ T_v(y) + T_{v-1}(x)\ T_v(y)$$

$$2y\ T_v(y)\ T_v(x) = T_{v+1}(y)\ T_v(x) + T_{v-1}(y)\ T_v(x).$$

If one subtracts the latter equation from the former equation, the following equation is obtained

$$2(x-y)\ T_v(x)\ T_v(y) = T_{v+1}(x)\ T_v(y) - T_{v-1}(y)\ T_v(x)$$
$$+ T_{v-1}(x)\ T_v(y) - T_{v+1}(y)\ T_v(x).$$

Summing both sides of this equation from $v = 1$ to $v = n$, simplifying the right-hand side and dividing each term by $x-y$, one obtains the desired result.

Theorem 4.5. The sum of the first $n+1$ Tchebycheff polynomials can be expressed in terms of Tchebycheff polynomials in the following manner

$$\sum_{v=0}^{n} T_v(x) = \frac{T_{n+1}(x) - T_n(x) + x - 1}{2(x - 1)} .$$

Proof: From the Definition 4.1 it is easily seen that $T_r(1) = 1$. Setting $y = 1$ in the equation of Theorem 4.4, adding 1 to both sides of the equation and dividing each side of the equation by 2, one obtains the desired result.

The replacing of x by $T_p(x)$ and of y by $T_q(y)$ in the equations of Theorems 4.4 and 4.5 yields the following results.

$$1 + 2\sum_{v=1}^{n} T_{vp}(x)\, T_{vq}(y) = \frac{T_{(n+1)p}(x)\, T_{nq}(y) - T_{np}(x)\, T_{(n+1)q}(y)}{T_p(x) - T_q(y)},$$

$$\sum_{v=0}^{n} T_{vp}(x) = \frac{T_{(n+1)p}(x) - T_{np}(x) + T_p(x) - 1}{2\left[T_p(x) - 1\right]} .$$

Here the first equation is seen to be a generalization of the well known Christoffel relation for Tchebycheff polynomials. It is obvious how these results may be generalized further.

C. A Theorem on Series Involving Products of Tchebycheff Polynomials

Theorem 4.6. For $|t| < 1$,

$$\sum_{n=0}^{\infty} t^n\, T_n(x)\, T_n(y)\, T_n(z) =$$

$$\frac{1 - 7xyzt + 3At^2 + 5Bxyzt^3 + Ct^4 + 3Bxyzt^5 + At^6 - xyzt^7}{1 - 8xyzt + 4At^2 + 8Bxyzt^3 + 2Ct^4 + 8Bxyzt^5 + 4At^6 - 8xyzt^7 + t^8}$$

<u>where</u> $A = 1-2y^2-2x^2-2z^2+4x^2z^2+4x^2y^2+4y^2z^2$

$B = 5-4x^2-4y^2-4z^2$

$C = 3-8y^2-8z^2-8x^2+8y^4+8z^4+8x^4+32x^2y^2z^2$.

<u>Proof:</u> It is well known (8, p. 80) that a generating function for $T_n(z)$ is $\frac{1-t^2}{1-2tz+t^2}$, that is,

$$\frac{1-t^2}{1-2tz+t^2} = \sum_0^{\infty} \epsilon_n \, T_n(z) \, t^n. \qquad |t|<1 \tag{4.4}$$

Adding 1 to both sides of this equation and dividing both sides by 2, one obtains the following equation

$$\frac{1-zt}{1-2zt+t^2} = \sum_0^{\infty} T_n(z) \, t^n.$$

The setting of $z = \cos w$ yields

$$\frac{1-t\cos w}{1-2t\cos w+t^2} = \sum_0^{\infty} t^n \cos n\,w.$$

Setting first w equal to x_1+x_2 and then equal to x_1-x_2 , and adding the two resulting equations, one obtains the following equation

$$\sum_0^{\infty} 2t^n \cos nx_1 \cos nx_2$$

$$= \frac{1-t\cos(x_1+x_2)}{1-2t\cos(x_1+x_2)+t^2} + \frac{1-t\cos(x_1-x_2)}{1-2t\cos(x_1-x_2)+t^2} \tag{4.5}$$

which with $x_1 = \text{Arc}\cos x$, $x_2 = \text{Arc}\cos y$ becomes

$$\sum_0^\infty 2t^n\, T_n(x)\, T_n(y)$$

$$= \frac{1-t(xy-\sqrt{1-x^2}\sqrt{1-y^2})}{1-2t(xy-\sqrt{1-x^2}\sqrt{1-y^2})+t^2} + \frac{1-t(xy+\sqrt{1-x^2}\sqrt{1-y^2})}{1-2t(xy+\sqrt{1-x^2}\sqrt{1-y^2})+t^2}$$

since $\cos(x_1+x_2) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$ and $\cos(x_1-x_2)=xy+\sqrt{1-x^2}\sqrt{1-y^2}$. Rationalizing the denominators on the right-hand side of the above equation, adding the two resulting expressions, and dividing both sides of the equation by 2, one obtains the following equation

$$\sum_0^\infty t^n\, T_n(x)\, T_n(y) = \frac{1-3txy+t^2(2x^2+2y^2-1) - t^3xy}{1-4txy + 2t^2(2x^2+2y^2-1) -4t^3xy+t^4}.$$

The desired result may then be obtained if one sets x equal to x_3+x_4 and x_3-x_4 successively in equation (4.5), adds the two resulting expressions, replaces x_3 by Arc cos x, x_4 by Arc cos y, and x_2 by Arc cos z, and simplifies the right-hand side of the obtained equation in a manner similar to that above.

The above theorem can be generalized to the infinite sums of products of p Tchebycheff polynomials by the use of mathematical induction. One has, in fact,

$$\sum_0^\infty 2^{p-1}\, t^n\, T_n(x_1)\, T_n(x_2) \,\ldots\, T_n(x_p)$$

$$= \sum_{j_2,\ldots,j_p} \frac{1-t\cos(x_1+j_2x_2+j_3x_3+\ldots+j_px_p)}{1-2t\cos(x_1+j_2x_2+j_3x_3+\ldots+j_px_p)+t^2} \qquad (4.6)$$

where the summation denotes the sum of $j_2,\ldots,j_p$, over all possible combinations of +1 and -1.

As in the previous section one can obviously generalize the result of Theorem 4.6, or more generally equation (4.6), by the use of commutativity. Also if one sets $x=y=z=t$ in Theorem 4.6, the following theorem is obtained.

<u>Theorem</u> 4.7. <u>For</u> $|t|<1$,

$$\sum_{0}^{\infty} t^n\, T_n^3(t) = \frac{7t^2+1}{8t^2+1}\,.$$

V. APPLICATION OF THE VOLTERRA TRANSFORM TO TCHEBYCHEFF POLYNOMIALS

A. Some Theorems on Tchebycheff Polynomials

The Neumann polynomial $O_n(u)$ is defined (8, p. 38) by

$$\frac{1}{u-z} = \sum_{n=0}^{\infty} \epsilon_n\, J_n(z)\, O_n(u).$$

$O_n(u)$ is a polynomial in powers of $\frac{1}{u}$. If in this polynomial, u is replaced by $\frac{1}{z}$, a polynomial in powers of z results. It can easily be seen that the Volterra transform of $O_n(\frac{1}{z})$ is i^{n-1} $T_n(y-x)$, that is,

$$V\left[i^{n+1}\, O_n(\tfrac{1}{z})\right] = T_n(y-x).$$

It is known (8, p. 38) that

$$(n-1)\, O_{n+1}(t) + (n+1)\, O_{n-1}(t) = \frac{2(n^2-1)}{t}\, O_n(t) + \frac{2n}{t} \sin^2 \frac{n\pi}{2}. \quad (5.1)$$

<u>Theorem</u> 5.1. <u>The Tchebycheff polynomials satisfy the following equation</u>

$$\frac{T_{n+1}(y-x)}{n+1} - \frac{T_{n-1}(y-x)}{n-1} = 2\int_x^y T_n(t-x)\,dt + \frac{2n i^{n-1}}{n^2-1} \sin^2 \frac{n\pi}{2}$$

<u>for</u> $n > 1$.

Proof: The theorem is easily proved if one sets $t = \frac{1}{z}$ in equation (5.1), takes the Volterra transform of each term, and divides each term by n^2-1.

Theorem 5.2. The Tchebycheff polynomials satisfy the following equation

$$\frac{T_{p+1}(y-x)}{p+1} + \frac{T_p(y-x)}{p} - \frac{T_2(y-x)}{2} - T_1(y-x)$$

$$= 2\sum_{n=2}^{p}\int_x^y T_n(t-x)dt + 2\sum_{n=2}^{p}\frac{ni^{n-1}}{n^2-1}\sin^2\frac{n\pi}{2},$$

for $p \geq 2$.

Proof: If both sides of the equation in Theorem 5.1 are summed from 2 to p, the following equation is obtained

$$\sum_{n=2}^{p}\left[\frac{T_{n+1}(y-x)}{n+1} - \frac{T_{n-1}(y-x)}{n-1}\right] = 2\sum_{n=2}^{p}\int_x^y T_n(t-x)dt + 2\sum_{n=2}^{p}\frac{ni^{n-1}}{n^2-1}\sin^2\frac{n\pi}{2},$$

which through cancellation of like terms on the left-hand side of the above equation gives the desired result.

Theorem 5.3. The Tchebycheff polynomials and the hypergeometric series ${}_2F_1$ satisfy the following equation

$$\int_x^y \frac{T_n(\sqrt{1-(t-x)}\,)\,T_n(\sqrt{1-(t-y)}\,)dt}{\sqrt{t-x}\,\sqrt{y-t}\,\sqrt{1-(t-x)}\,\sqrt{1-(t-y)}} = \pi\,{}_2F_1\left[\frac{1+n}{2}, \frac{1-n}{2}; 1; (y-x)^2\right].$$

Proof: If in the equation in Theorem 3.2, p is set equal to $\frac{a+b}{2}$, then

$$\int_x^y \left[(t-x)(y-t)\right]^{\frac{a+b}{2}-1} {}_2F_1\left[a,b;\tfrac{a+b}{2};(t-x)\right] {}_2F_1\left[a,b;\tfrac{a+b}{2};t-y\right] dt$$

$$= \frac{\Gamma^2(\frac{a+b}{2})}{\Gamma(a+b)}(y-x)^{a+b-1}\ {}_2F_1\left[a,b;a+b;(y-x)^2\right]. \tag{5.2}$$

It is well known (8, p. 8) that

$$\cos n z = \cos z\ {}_2F_1\ (\tfrac{1+n}{2},\ \tfrac{1-n}{2};\ \tfrac{1}{2};\ \sin^2 z).$$

Setting $z = \text{Arc} \cos (y-x)$, one obtains

$$T_n(y-x) = (y-x)\ {}_2F_1\left[\tfrac{1+n}{2},\ \tfrac{1-n}{2};\ \tfrac{1}{2};\ 1-(y-x)^2\right],$$

or

$$\frac{T_n\ (\sqrt{1-(y-x)}\ )}{\sqrt{1-(y-x)}} = {}_2F_1\left[\tfrac{1+n}{2},\ \tfrac{1-n}{2};\ \tfrac{1}{2};\ (y-x)\right]. \tag{5.3}$$

Setting $a = \frac{1+n}{2}$ and $b = \frac{1-n}{2}$ in equation (5.2), one obtains the following equation

$$\int_x^y \frac{{}_2F_1\left[\frac{1+n}{2},\frac{1-n}{2};\frac{1}{2};t-x\right]}{\sqrt{t-x}}\ \frac{{}_2F_1\left[\frac{1+n}{2},\frac{1-n}{2};\frac{1}{2};t-y\right]dt}{\sqrt{y-t}} = \pi\,{}_2F_1\left[\tfrac{1+n}{2},\tfrac{1-n}{2};1;(y-x)^2\right],$$

which together with equation (5.3) gives the desired result.

B. The Polynomials ${}_2F_2(n,\ -n;\ \frac{1}{2},\ p;\ z)$

Since $V\left[z^p\ T_n(1-2z)\right] = \frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_2F_2(n,\ -n;\ \frac{1}{2},\ p;\ y-x)$, $p > 0$, many properties of the polynomials ${}_2F_2(n,\ -n;\ \frac{1}{2},\ p;z)$, $p > 0$, can be derived from the identities and theorems concerning Tchebycheff polynomials in Chapter IV by the use of the Volterra transform. The following theorems serve to illustrate the method used in deriving some properties of these polynomials.

<u>Theorem</u> 5.4. <u>The hypergeometric series</u> ${}_2F_2$ <u>satisfy the following integral addition formula</u>

$$2\int_x^y \frac{[(t-x)(y-t)]^{p-1}}{\Gamma(p)\,\Gamma(p)}\ {}_2F_2(n,-n;\tfrac{1}{2},p;y-t)\ {}_2F_2(m,-m;\tfrac{1}{2},p;t-x)\,dt$$

$$=\frac{(y-x)^{2p-1}}{\Gamma(2p)}\,{}_2F_2(n-m,m-n;\tfrac{1}{2},2p;y-x)+\frac{(y-x)^{2p-1}}{\Gamma(2p)}\,{}_2F_2(n+m,-n-m;\tfrac{1}{2},2p;y-x),$$

<u>for</u> $p>0$.

<u>Proof</u>: Setting $x = 1-2z$ in equation (4.1) and multiplying both sides of the resulting equation by z^{2p}, $p>0$, one obtains the following equation

$$2\left[z^p\ T_n(1-2z)\right]\cdot\left[z^p\ T_m(1-2z)\right] = z^{2p}\ T_{n+m}(1-2z)+z^{2p}\ T_{n-m}(1-2z).$$

Taking the Volterra transform of this equation with the left-hand term factored in the above indicated manner, one obtains the following equation

$$2\left[\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_2F_2(n,-n;\tfrac{1}{2},p;y-x)\right]*\left[\frac{(y-x)^{p-1}}{\Gamma(p)}\ {}_2F_2(m,-m;\tfrac{1}{2},p;y-x)\right]$$

$$=\frac{(y-x)^{2p-1}}{\Gamma(2p)}\ {}_2F_2(n+m\ -n-m;\tfrac{1}{2},2p;y-x)+\frac{(y-x)^{2p-1}}{\Gamma(2p)}\,{}_2F_2(n-m,m-n;\tfrac{1}{2},2p;y-x),$$

which is the desired result.

Theorem 5.5. The generating function for the polynomials

$${}_2F_2(n,-n;\tfrac{1}{2},p;z)$$

is

$$\frac{1+t}{1-t}\,{}_1F_1\left[1;p;\frac{-4tz}{(1-t)^2}\right],$$

that is

$$\frac{1+t}{1-t}\,{}_1F_1\left[1;p;\frac{-4tz}{(1-t)^2}\right]=\sum_0^\infty \epsilon_n\;{}_2F_2(n,-n;\tfrac{1}{2},p;z)t^n.$$

Proof: Setting $x = 1-2z$ and multiplying each side of equation (4.4) by z^p, $p > 0$, one obtains the following equation

$$\frac{(1-t^2)z^p}{(1-t)^2+4tz}=\sum_0^\infty \epsilon_n\; z^p\; T_n(1-2z)t^n.$$

Expanding the denominator on the left-hand side in powers of z and taking the Volterra transform of each side one obtains the following result.

$$\frac{(1+t)}{(1-t)}\;\frac{(y-x)^{p-1}}{\Gamma(p)}\;{}_1F_1\left[1;p;\frac{-4t(y-x)}{(1-t)^2}\right]$$

$$=\sum_0^\infty \epsilon_n\;\frac{(y-x)^{p-1}}{\Gamma(p)}\;{}_2F_2(n,-n;\tfrac{1}{2},p;y-x)t^n.$$

The division of both sides of this equation by $\frac{(y-x)^{p-1}}{\Gamma(p)}$ and the setting of $(y-x) = z$ then yields the desired result.

Theorem 5.6. The sum of the first $n+1$ polynomials of the type ${}_2F_2(v,\ -v;\ \frac{1}{2},\ p+1;\ z)$ can be expressed in terms of these polynomials in the following manner.

$$-4\sum_{v=0}^{n}\frac{z}{p}\;{}_2F_2(v,-v;\tfrac{1}{2},p+1;z)$$

$$={}_2F_2(n+1,-n-1;\tfrac{1}{2},p;z)\;-\;{}_2F_2(n,-n;\tfrac{1}{2},p;z)\;-\;\frac{2z}{p}.$$

Proof: Replacing x by 1-2z in the equation in Theorem 4.5 and multiplying each term by z^p, $p>0$, one obtains the following equation

$$-4\, z^{p+1} \sum_{v=0}^{n} T_v(1-2z) = z^p\, T_{n+1}(1-2z) - z^p\, T_n(1-2z) - 2z^{p+1}.$$

Taking the Volterra transform of each term of this equation, one finds that

$$-4 \sum_{v=0}^{n} \frac{(y-x)^p}{\Gamma(p+1)}\, {}_2F_2(v,-v;\tfrac{1}{2},p+1;y-x)$$

$$=\frac{(y-x)^{p-1}}{\Gamma(p)}\, {}_2F_2(n+1,-n-1;\tfrac{1}{2},p;y-x)-\frac{(y-x)^{p-1}}{\Gamma(p)}\, {}_2F_2(n,-n;\tfrac{1}{2},p;y-x)-\frac{2(y-x)^p}{\Gamma(p+1)}.$$

Dividing this equation by $\frac{(y-x)^{p-1}}{\Gamma(p)}$ and setting $y-x=z$, one sees that the desired result is obtained.

Theorem 5.7. The hypergeometric series ${}_2F_2$ satisfy the following integral addition formula

$$\sum_{n=1}^{\infty} \frac{(a+b)^n (y-x)^{n-1}}{n!\,\Gamma(n)2^n} \sum_{k=0}^{[\frac{n}{2}]} \epsilon_{[\frac{n+1}{2}]-k} \binom{n}{k}\, {}_2F_2\left[(n-2k)p,-(n-2k)p;\tfrac{1}{2};n;y-x\right]$$

$$=\sum_{n=1}^{\infty} \frac{a^n}{n!} \frac{(y-x)^{n-1}}{\Gamma(n)2^n} \sum_{k=0}^{[\frac{n}{2}]} \epsilon_{[\frac{n+1}{2}]-k} \binom{n}{k}\, {}_2F_2\left[(n-2k)p,-(n-2k)p;\tfrac{1}{2};n;y-x\right]$$

$$+\sum_{n=1}^{\infty} \frac{b^n}{n!} \frac{(y-x)^{n-1}}{\Gamma(n)2^n} \sum_{k=0}^{[\frac{n}{2}]} \epsilon_{[\frac{n+1}{2}]-k} \binom{n}{k}\, {}_2F_2\left[(n-2k)p,-(n-2k)p;\tfrac{1}{2},n;y-x\right]$$

$$+\int_x^y \sum_{n=1}^{\infty} \frac{a^n}{n!} \frac{(t-x)^{n-1}}{\Gamma(n)2^n} \sum_{k=0}^{[\frac{n}{2}]} \epsilon_{[\frac{n+1}{2}]-k} \binom{n}{k}\, {}_2F_2\left[(n-2k)p,-(n-2k)p;\tfrac{1}{2},n;t-x\right]$$

$$\cdot\sum_{m=1}^{\infty} \frac{b^m}{m!} \frac{(y-t)^{m-1}}{\Gamma(m)2^m} \sum_{k=0}^{[\frac{m}{2}]} \epsilon_{[\frac{m+1}{2}]-k} \binom{m}{k}\, {}_2F_2\left[(m-2k)p,-(m-2k)p;\tfrac{1}{2},m;y-t\right]dt.$$

<u>Proof:</u> If in the equation in Theorem 4.3, x is replaced by (1-2z) and the resulting equation is multiplied by z^u, then the following equation is obtained

$$2^u\left[zT_p(1-2z)\right]^u = \sum_{k=0}^{[\frac{u}{2}]} \epsilon_{[\frac{u+1}{2}]-k} \binom{u}{k} z^u\, T_{(u-2k)p}(1-2z).$$

Taking the Volterra transform of this equation one obtains the following result

$$2^u\left\{V\left[z\; T_p(1-2z)\right]\right\}^{*u} = \sum_{k=0}^{[\frac{u}{2}]} \epsilon_{[\frac{u+1}{2}]-k} \binom{u}{k}\, V\left[z^u\; T_{(u-2k)p}(1-2z)\right],$$

which by the table of transforms becomes

$$2^u\; {}_2F_2^{\;*u}(p,-p;\tfrac{1}{2},1;y-x)$$

$$= \sum_{k=0}^{[\frac{u}{2}]} \epsilon_{[\frac{u+1}{2}]-k} \binom{u}{k} \frac{(y-x)^{u-1}}{\Gamma(u)} \cdot {}_2F_2\left[(u-2k)p,-(u-2k)p;\tfrac{1}{2},u;y-x\right].$$

If $f(x,y) = {}_2F_2(p,-p;\frac{1}{2},1;y-x)$ then the Volterra transcendental $V(a,f^*)$ becomes

$$\sum_{n=1}^{\infty} \frac{a^n}{n!}\, {}_2F_2^{\;*n}(p,-p;\tfrac{1}{2},1;y-x).$$

Thus by using the above expression for ${}_2F_2^{\;*u}(p,-p;\frac{1}{2},1;y-x)$ one establishes the fact that

$$V\left[a,\; {}_2F_2(p,-p;\tfrac{1}{2},1;y-x)\right]$$

$$= \sum_{n=1}^{\infty} \frac{a^n}{n!} \frac{1}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \epsilon_{[\frac{n+1}{2}]-k} \binom{n}{k} \frac{(y-x)^{n-1}}{\Gamma(n)} \cdot {}_2F_2\left[(n-2k)p,-(n-2k)p;\tfrac{1}{2},n;y-x\right].$$

Restating the integral addition formula (2.3) with $f(x,y) = {}_2F_2(p,-p;\frac{1}{2},1;y-x)$ using the above expression for $V\left[a,\ {}_2F_2(p,-p;\frac{1}{2},1;y-x)\right]$, one obtains the desired result.

C. Integral Addition Theorems of Bessel Functions

Definition 5.1. The Tchebycheff transcendental is defined as $e^{-af^*} f^{*u}\ {}_2F_1(n,-n;\frac{1}{2};\ f^*)$, $u>0$, and is designated by $T\left[n,a,u;f^*\right]$.

Writing z in place of $\frac{1-x}{2}$ in equation (4.2) and multiplying both sides of the resulting equation by $e^{-(a+b)z}\ z^{u+v}$, $u>0$, $v>0$, one obtains the following equation

$$2\left[e^{-az}\ z^u\ {}_2F_1(n,-n;\tfrac{1}{2};z)\right]\left[e^{-bz}\ z^v\ {}_2F_1(m,-m;\tfrac{1}{2};z)\right]$$
$$= e^{-(a+b)z}\ z^{u+v}{}_2F_1(n-m,m-n;\tfrac{1}{2};z)+e^{-(a+b)z}\ z^{u+v}{}_2F_1(n+m,-n-m;\tfrac{1}{2};z).$$

With the arrangement of terms as indicated, by replacing powers of z by powers by composition of f(x,y), one obtains the following integral addition formula

$$2\left[e^{-af^*}f^{*u}{}_2F_1(n,-n;\tfrac{1}{2};f^*)\right]^*\left[e^{-bf^*}f^{*v}{}_2F_1(m,-m;\tfrac{1}{2};f^*)\right]$$
$$= e^{-(a+b)f^*}f^{*u+v}{}_2F_1(n-m,m-n;\tfrac{1}{2};f^*)$$
$$+ e^{-(a+b)f^*}f^{*u+v}{}_2F_1(n+m,-n-m;\tfrac{1}{2};f^*). \qquad (5.4)$$

This integral addition formula may be written in terms of the Tchebycheff transcendental as follows

$$2T\left[n,a,u;f^*\right]^*T\left[m,b,v,f^*\right]=T\left[n-m,a+b,u+v;f^*\right]+T\left[n+m,a+b,u+v;f^*\right]. \quad (5.5)$$

<u>Theorem</u> 5.8. <u>The Bessel functions satisfy the following integral addition theorem</u>

$$2nm\sqrt{\pi}\int_x^y \sum_{i=0}^{n} \frac{\Gamma(n+i)(-1)^i}{\Gamma(n-i+1)\Gamma(\frac{1}{2}+i)\ i!} \left(\frac{t-x}{a}\right)^{\frac{u+i-1}{2}} J_{u+i-1}\left[2\sqrt{a(t-x)}\right]$$

$$\cdot \sum_{j=0}^{m} \frac{\Gamma(m+j)(-1)^j}{\Gamma(m-j+1)\Gamma(\frac{1}{2}+j)\ j!} \left(\frac{y-t}{b}\right)^{\frac{v+j-1}{2}} J_{v+j-1}\left[2\sqrt{b(y-t)}\right]dt$$

$$= \sum_{i=0}^{n+m} \frac{\Gamma(n+m+i)(-1)^i\ (n+m)}{\Gamma(n+m-i+1)\Gamma(\frac{1}{2}+i)\ i!} \left(\frac{y-x}{a+b}\right)^{\frac{u+v+i-1}{2}} J_{u+v+i-1}\left[2\sqrt{(a+b)(y-x)}\right]$$

$$+ \sum_{i=0}^{n-m} \frac{\Gamma(n-m+i)(-1)^i\ (n-m)}{\Gamma(n-m-i+1)\Gamma(\frac{1}{2}+i)\ i!} \left(\frac{y-x}{a+b}\right)^{\frac{u+v+i-1}{2}} J_{u+v+i-1}\left[2\sqrt{(a+b)(y-x)}\right]$$

<u>for</u> $n\neq 0$, $m\neq 0$, $u>0$, $v>0$.

<u>Proof</u>: The Tchebycheff transcendental $T[n,a,u;1^*]$ is equal to $e^{-a1^*}\ 1^{*u}\ {}_2F_1(n,-n;\frac{1}{2};1^*)$. However,

$$e^{-a1^*}\ 1^{*u}\ {}_2F_1(n,-n;\tfrac{1}{2};1^*) = \sum_{i=0}^{n} \frac{\Gamma(n+i)n\sqrt{\pi}(-1)^i\ e^{-a1^*}\ 1^{*u+i}}{\Gamma(n-i+1)\Gamma(\frac{1}{2}+i)\ i!}$$

which in turn is equal to

$$\sum_{i=0}^{n} \frac{\Gamma(n+i)\ n\sqrt{\pi}(-1)^i}{\Gamma(n-i+1)\Gamma(\frac{1}{2}+i)\ i!} \left(\frac{y-x}{a}\right)^{\frac{u+i-1}{2}} J_{u+i-1}\left[2\sqrt{a(y-x)}\right]$$

by the table of transforms. Restating the integral addition formula (5.4) with $f(x,y) = 1$, one obtains the desired result.

Special cases of the above integral addition theorem can be obtained if $n = m = 0$ or if either n or m is zero, but these will not be written out here. The case where $n = m = 0$ has been given previously in the literature by Thielman (12).

By defining a generalized Tchebycheff transcendental, the integral addition formula (5.4) can be extended. Let

$$T\left[n,a,u,r;f^*\right] = e^{-af^*} f^{*u} \; {}_2F_1(n,-n;\tfrac{1}{2};f^{*r}), \tag{5.6}$$

r 0, be the generalized Tchebycheff transcendental. It is clear that $T\left[n,a,u,r;f^*\right]$ satisfies an integral addition theorem similar to equation (5.4), namely the following one.

$$2T\left[n,a,u,r;f^*\right]^* \; T\left[m,b,v,r;f^*\right]$$

$$= T\left[n-m,a+b,u+v,r;f^*\right] + T\left[n+m,a+b,u+v,r;f^*\right]. \tag{5.7}$$

If $f(x,y) = 1$, the Tchebycheff transcendental takes the following form

$$T\left[n,a,u,r;1^*\right] = \sum_{i=0}^{n} \frac{\Gamma(n+i)\,n\sqrt{\pi}\,(-1)^i}{\Gamma(n-i+1)\Gamma(\frac{1}{2}+i)\,i\,!} \left(\frac{y-x}{a}\right)^{\frac{u+ir-1}{2}} J_{u+ir-1}\left[2\sqrt{a(y-x)}\right].$$

Thus, by setting $f(x,y) = 1$ in equation (5.7), one obtains the following generalization of the preceding integral addition theorem for Bessel functions.

<u>Theorem</u> 5.9. For $n \neq 0$, $m \neq 0$, $v > 0$, $r > 0$, $u > 0$,

$$2nm\sqrt{\pi}\int_x^y \sum_{i=0}^{n} \frac{\Gamma(n+i)(-1)^i}{\Gamma(n-i+1)\Gamma(\frac{1}{2}+i)\,i\,!} \left(\frac{t-x}{a}\right)^{\frac{u+ir-1}{2}} J_{u+ir-1}\left[2\sqrt{a(t-x)}\right]$$

$$\cdot \sum_{j=0}^{m} \frac{\Gamma(m+j)(-1)^j}{\Gamma(m-j+1)\Gamma(\frac{1}{2}+j)\,j\,!} \left(\frac{y-t}{b}\right)^{\frac{v+jr-1}{2}} J_{v+jr-1}\left[2\sqrt{b(y-t)}\right]dt$$

$$= \sum_{i=0}^{n+m} \frac{\Gamma(n+m+i)(n+m)(-1)^i}{\Gamma(n+m-i+1)\Gamma(\frac{1}{2}+i)\,i\,!} \left(\frac{y-x}{a+b}\right)^{\frac{u+ir+v-1}{2}} J_{u+ir+v-1}\left[2\sqrt{(a+b)(y-x)}\right]$$

$$+\sum_{i=0}^{n-m}\frac{\Gamma(n-m+i)(n-m)(-1)^i}{\Gamma(n-m-i+1)\Gamma(\frac{i}{2}+1)i!}\left(\frac{y-x}{a+b}\right)^{\frac{u+ir+v-1}{2}} J_{u+ir+v-1}\left[2\sqrt{(a+b)(y-x)}\right].$$

For r = 1 this theorem reduces to Theorem 5.8. For n=m=0 and for either n or m equal to zero special cases of the above theorem can be written out in a manner similar to those special cases of Theorem 5.8.

D. An Integral Addition Theorem Involving Laguerre Polynomials

Theorem 5.10. The Laguerre polynomials $L_n^p(y-x)$ satisfy the following integral addition theorem

$$2\int_x^y \sum_{i=0}^{n}\frac{n\sqrt{\pi}\,\Gamma(n+i)(-1)^i\, r!}{\Gamma(n+1-i)\Gamma(\frac{i}{2}+1)i!\,\Gamma(ip+k+r)}(t-x)^{ip+k-1}L_r^{ip+k-1}(t-x)$$

$$\cdot\sum_{j=0}^{m}\frac{m\sqrt{\pi}\,\Gamma(m+j)(-1)^j\, s!}{\Gamma(m+1-j)\Gamma(\frac{j}{2}+j)j!\,\Gamma(jp+q+s)}(y-t)^{jp+q-1}L_s^{jp+q-1}(y-t)\,dt$$

$$=\sum_{i=0}^{n+m}\frac{(n+m)\sqrt{\pi}\,\Gamma(n+m+i)(-1)^i(r+s)!}{\Gamma(\frac{i}{2}+1)\Gamma(n+m+1-i)i!\,\Gamma(k+q+ip+r+s)}(y-x)^{k+q+ip-1}L_{r+s}^{k+q+ip-1}(y-x)$$

$$+\sum_{i=0}^{n-m}\frac{(n-m)\sqrt{\pi}\,\Gamma(n-m+i)(-1)^i(r+s)!}{\Gamma(n-m-i+1)\Gamma(\frac{i}{2}+1)i!\,\Gamma(k+q+ip+r+s)}(y-x)^{k+q+ip-1}L_{r+s}^{k+q+ip-1}(y-x)$$

for all k,r,q,s, and p greater than zero.

Proof: Replacing $\frac{1-x}{2}$ by z^p in equation (4.2) and multiplying both sides of the resulting equation by $z^{k+q}(1-z)^{r+s}$, one obtains the following equation

$$2\left[z^k(1-z)^r\ {}_2F_1(n,-n;\tfrac{1}{2};z^p)\right]\cdot\left[z^q(1-z)^s\ {}_2F_1(m,-m;\tfrac{1}{2};z^p)\right]$$
$$= z^{k+q}(1-z)^{r+s}\ {}_2F_1(n+m,-n-m;\tfrac{1}{2};z^p)$$
$$+ z^{k+q}(1-z)^{r+s}\ {}_2F_1(n-m,m-n;\tfrac{1}{2};z^p). \qquad (5.8)$$

By definition

$$z^k(1-z)^r\ {}_2F_1(n,-n;\tfrac{1}{2};z^p) = \sum_{i=0}^{n} \frac{n\sqrt{\pi}\,\Gamma(n+i)(-1)^i z^{ip+k}(1-z)^r}{\Gamma(n+1-i)\Gamma(\frac{1}{2}+i)\ i\,!}\ .$$

By taking the Volterra transform of equation (5.8) with the left-hand term factored as indicated, one obtains the following equation

$$2\left[\sum_{i=0}^{n} \frac{n\sqrt{\pi}\,\Gamma(n+i)(-1)^i\ 1^{*ip+k}(1^{*o}-1^{*1})^r}{\Gamma(n+1-i)\Gamma(\frac{1}{2}+i)i\,!}\right]^*$$
$$\left[\sum_{j=0}^{m} \frac{m\sqrt{\pi}\,\Gamma(m+j)(-1)^j\ 1^{*jp+q}(1^{*o}-1^{*1})^s}{\Gamma(m+1-j)\Gamma(\frac{1}{2}+j)j\,!}\right]$$
$$= \sum_{i=0}^{n+m} \frac{(n+m)\sqrt{\pi}\,\Gamma(n+m+i)(-1)^i\ 1^{*k+q+ip}(1^{*o}-1^{*1})^{r+s}}{\Gamma(\frac{1}{2}+i)\Gamma(n+m+1-i)\ i\,!}$$
$$+ \sum_{i=0}^{n-m} \frac{(n-m)\sqrt{\pi}\,\Gamma(n-m+i)(-1)^i\ 1^{*k+q+ip}(1^{*o}-1^{*1})^{r+s}}{\Gamma(\frac{1}{2}+i)\Gamma(n-m+1-i)\ i\,!}, \qquad (5.9)$$

where 1^{*o} is defined so that $1^{*o*}1^{*a} = 1^{*a}$ (19, p. 102).

It can be shown that $1^{*a+1}(1^{*o}-1^{*1})^b = \dfrac{b\,!(y-x)^a\ L_b{}^a(y-x)}{\Gamma(a+b+1)}$.

Making use of this fact one can rewrite equation (5.9) in terms of Laguerre polynomials and obtain the desired result.

If n=m=0 or either n or m is zero then special cases of the above theorem can be written out in a manner similar to those special cases of the Bessel functions.

VI. SUMMARY

The main results of this thesis are the generalizations of previous known results given by Hadamard (6), and Thielman (12) on integral addition theorems of Bessel functions, namely Theorems 5.8 and 5.9. Also an integral addition theorem, Theorem 5.10, for Laguerre polynomials is obtained.

The methods used are based on the isomorphism which exists between Volterra's theory of permutable functions and the algebra of polynomials and power series. From known algebraic relations between certain given functions, integral addition theorems are obtained for new functions which are the Volterra transforms (see Definition 3.1) of the given functions. In particular recursion formulas for Tchebycheff polynomials lead to the integral addition theorems mentioned above.

Other applications of the theory of composition are given, some of these lead to the evaluation of certain integrals and series expansions for hypergeometric functions. Identities concerning Tchebycheff polynomials are derived on the basis of the commutative property (see Definition 4.2) of these polynomials. Also an expression for a series involving triple products of Tchebycheff polynomials is obtained directly from the generating function of these polynomials. Since the Volterra transform of $z^p\, T_n(1-2z)$ is ${}_2F_2(n,-n;\frac{1}{2},p;y-x)$, $p>0$ some properties of the set of polynomials ${}_2F_2(n,-n;\frac{1}{2},p;t)$, $(n=0,1,2,\ldots)$, are obtained from these identities and theorems on Tchebycheff polynomials.

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